Finite-Memory Least Squares Universal Prediction of Individual Continuous Sequences

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***Abstract*—In this paper we consider the problem of universal prediction of individual *continuous* sequences with square-error loss, using a deterministic finite-state machine (FSM). The goal is to attain universally the performance of the best constant predictor tuned to the sequence, which predicts the empirical mean and incurs the empirical variance as the loss. The paper analyzes the tradeoff between the number of states of the universal FSM and the excess loss (regret). We first present the Exponential Decaying Memory (EDM) machine, used in the past for predicting binary sequences, and show bounds on its performance. Then we look explicitly for the optimal machine with a small number of states. We consider a class of machines denoted the Degenerated Tracking Memory (DTM) machines that outperform the EDM machine for a small number of states. Unfortunately, the regret of the DTM machines is bounded away from zero even with a large number of states making these machines suboptimal. Finally, we prove a lower bound on the achievable regret of any FSM that defines the best rate that the regret can vanish. We propose a new machine, the Enhanced Exponential Decaying Memory, which attains the bound and outperforms the EDM for any number of states.**

***Index Terms*—Universal prediction, individual continuous sequences, finite-memory, least-squares.**

I. Introduction

Consider a continuous-valued *individual* sequence $x_1, x_2, \ldots, x_t, \ldots$, where each sample is assumed to be bounded in the interval $[a, b]$ but otherwise arbitrary with no underlying statistics. At each time t, after observing x_1^t, a predictor guesses the next outcome $\hat{x}_{t+1}$, and incurs a square error prediction loss $(x_{t+1} - \hat{x}_{t+1})^2$. Suppose one can tune a (non-universal) predictor to the sequence, from a given class of predictors. For example, the best constant predictor for a given sequence, i.e. a predictor that uses a constant prediction for all the sequence outcomes, is the empirical mean $\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t$. The square error loss incurred by this predictor is the sequence's empirical variance $\frac{1}{n}\sum_{t=1}^{n}(x_t - \bar{x})^2$. Thus, for a given sequence x_1^n, the excess loss of a universal predictor U (that predicts $\hat{x}_{u,1}, ..., \hat{x}_{u,n}$) over the best constant predictor is termed the regret of the sequence w.r.t U:

$$R(U, x_1^n) = \frac{1}{n}\sum_{t=1}^{n}(x_t - \hat{x}_{u,t})^2 - \frac{1}{n}\sum_{t=1}^{n}(x_t - \bar{x})^2. \quad (1)$$

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In the individual setting, we analyze the performance of a universal predictor by the maximal excess loss, that is, the incurred regret of the worst sequence. An extensive survey on universal prediction is given in [1]. Some aspects of the universal prediction problem for individual *continuous* sequences with square error loss were already explored by Merhav and Feder in [2]. That work actually considered a more general case and showed that, e.g., the Recursive Least Squares (RLS) algorithm [3], [4] generates a universal predictor that attains the performance of the best (non-universal) L-order linear predictor [5] tuned to the sequence. When specialized to the "zero-order" case, i.e. the case where the non-universal predictor is the constant empirical mean predictor, the resulting universal predictor is the Cumulative Moving Average (CMA):

$$\hat{x}_{t+1} = (1 - \frac{1}{t+1})\hat{x}_t + \frac{1}{t+1}x_t, \quad (2)$$

where $\hat{x}_t$ is the prediction at time t. The regret of this predictor tends to zero with the sequence length n.

Note that while the reference non-universal constant predictor needs a single state, the universal predictor (2) requires an ever growing amount of memory. What happens if the universal predictor is also constrained to be a finite k-state machine? Universal estimation and prediction problems where the estimator/predictor is a k-state machine have been explored extensively in the past years. Cover [6] studied hypothesis testing problem where the tester has a finite memory. Hellman [7] studied the problem of estimating the mean of a Gaussian (or more generally stochastic) sequence using a finite state machine. This problem is closely related to our problem and may be considered as a stochastic version of it: if one assumes that the data is Gaussian than predicting it with a minimal mean square error essentially boils to estimating its mean. More recently, the finite-memory universal prediction problem for individual *binary* sequences with various loss functions was explored thoroughly in [8]–[13]. The finite-memory universal portfolio selection problem (that dealt with continuous-valued sequences but considered a very unique loss function) was also explored recently [14]. Yet, the basic problem of finite-memory universal prediction of *continuous-valued, individual* sequences with square error loss was left unexplored so far. This paper provides a solution for this problem, presenting such universal predictors attaining a vanishing regret when a large memory is allowed, but also maintaining an optimal tradeoff between the regret and the number of states used by the universal predictor.

The outline of the paper is as follows. In section II we formulate the discussed problem and present guidelines that will be used throughout this paper. In section III we propose a possible universal machine - the Exponential Decaying Memory (EDM) machine - and prove asymptotic lower and upper bounds on its worst regret. Section IV is devoted to universal prediction with a small number of states. We present the class of the Degenerated Tracking Memory (DTM) machines, an algorithm for designing the optimal DTM machine and a lower bound on the achievable regret. Sections V and VI are devoted to universal prediction using a large number of states. In section V we present an asymptotic lower bound on the achievable regret of any deterministic k-states machine and in section VI we present a new machine named the Enhanced Exponential Decaying Memory (E-EDM) machine that can attain any vanishing desired regret while outperforming the EDM machine. In section VII we summarize the results and suggest further research.

II. Definitions

Finite-state machine (FSM) is a commonly used model for sequential machines with a limited amount of storage. In our work we focus on time-invariant FSM.

Definition 1: A deterministic finite-state machine is defined by:

- An array of k states where $\{S_1, \ldots, S_k\}$ denote the value assigned to each state.
- The prediction of the machine at time t, $\hat{x}_t$, is the value assigned to the current state.
- The transition of the machine between states is defined by the maximum up and down steps from each state i, denoted $m_{u,i}$ and $m_{d,i}$ correspondingly, and by a threshold set $\{T_{i,-m_{d,i}-1}, T_{i,-m_{d,i}}, \ldots, T_{i,m_{u,i}-1}, T_{i,m_{u,i}}\}$ for each state i. Thus, if at time t the machine is at state i, it jumps j states ($-m_{d,i} \leq j \leq m_{u,i}$) if the input sample x_t satisfies $T_{i,j-1} \leq x_t < T_{i,j}$. Note that the thresholds are non-intersecting, where the union of them covers the interval $[a, b]$ (each input sample is assumed to be bounded in $[a, b]$).
- Equivalently, a transition function, $\varphi(i, x)$, can be defined for each state i where x is the input sample:

$$\varphi(i,x) = \begin{cases} i - m_{d,i} & , T_{i,-m_{d,i}-1} \leq x < T_{i,-m_{d,i}} \\ i - m_{d,i} + 1 & , T_{i,-m_{d,i}} \leq x < T_{i,-m_{d,i}+1} \\ \vdots & \\ i + m_{u,i} - 1 & , T_{i,m_{u,i}-2} \leq x < T_{i,m_{u,i}-1} \\ i + m_{u,i} & , T_{i,m_{u,i}-1} \leq x < T_{i,m_{u,i}} \end{cases}$$

Throughout this paper we discuss predictors designed for input samples that are bounded in $[0, 1]$. It is easily notable that any FSM designed to achieve regret smaller than R for any sequence bounded in $[0, 1]$, can be transformed into a FSM that achieves regret smaller than $(b-a)^2 R$ for any sequence bounded in $[a, b]$, where $a, b \in \mathbb{R}$, by applying a simple transformation - each state value S_i is transformed into $a + (b-a)S_i$ and each thresholds set $\underline{T}_{\ i}$ into $a + (b-a)\underline{T}_{\ i}$. Thus, all the results presented in this paper can be expanded to the more general case, where each individual sequence is assumed to be bounded in $[a, b]$.

We further present a theorem that will use us throughout this paper.

Definition 2: A circle is a cyclic closed set of L states\predictions $\{\hat{x}_t\}_{t=1}^{L}$, if there are input samples $\{x_t\}_{t=1}^{L}$ that rotate the machine between these states. A minimal circle is a circle that does not contain the same state more than once. An example is depict in Figure 1.

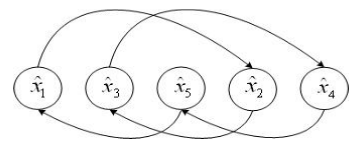

Fig. 1. Five states minimal circle. Arrows represent the jump at each time $t = 1, .., 5$, induced by a sequence of five samples.

Theorem 1: The worst sequence for a given FSM takes the machine to a minimal circle and rotates in it endlessly.

Proof: The proof is given in details in [15] where it is shown that the worst *binary* sequence for a given FSM w.r.t the log-loss function endlessly rotates the machine in a minimal circle. The proof is based on partitioning any long enough prediction sequence $\hat{x}_1, ..., \hat{x}_n$ of a given FSM for an input sequence $x_1, ...x_n$ into minimal circles and a negligible residual monotonic sequence. By using the convexity of the log-loss function, the regret over the entire sequence is upper bounded by the weighted average of the regrets of these minimal circles. Therefore a sequence that endlessly rotates the machine in the minimal circle with the highest regret will have a greater regret. Concluding that the worst sequence endlessly rotates the machine in a minimal circle. The proof for our case, the worst *continuous* sequence w.r.t the square loss function, is identical. ■

Note that there is an infinite number of sequences that can rotate a machine in a minimal circle (any input sample x that satisfies $T_{i,j-1} \leq x < T_{i,j}$ induces a j states jump from state i). In this paper we will refer the regret of a minimal circle as the regret of the sequence that endlessly rotates the machine in the minimal circle and achieve the highest regret.

III. The Exponential Decaying Memory machine

In [16] the Exponential Decaying Memory (EDM) machine has been presented as a universal predictor for individual *binary* sequences. It was further shown that the EDM machine with k states achieves an asymptotic regret of $O(k^{-2/3})$ compared to the constant predictors class w.r.t the log-loss (code length) and square-error functions.

We start by describing and adjusting the EDM machine for our case, predicting individual *continuous* sequences:

Definition 3: The ***Exponential Decaying Memory*** machine is defined by:

- k states $\{S_1, ..., S_k\}$ distributed uniformly over $[k^{-1/3}, 1 - k^{-1/3}]$.
- The transition function between states satisfies:

$$\hat{x}_{t+1} = Q(\hat{x}_t(1 - k^{-2/3}) + x_t k^{-2/3}) \ , \tag{3}$$

where $\hat{x}_t$ is the prediction (state) at time t and Q is the quantization function to the nearest state.

Note that the spacing gap between states, denoted Δ, satisfies:

$$\Delta = \tfrac{1-2k^{-1/3}}{k-1} \sim k^{-1} \ , \tag{4}$$

and the quantization function satisfies $Q(y) = \hat{x}_{t+1}$, if y satisfies $\hat{x}_{t+1} - \frac{1}{2}\Delta \leq y < \hat{x}_{t+1} + \frac{1}{2}\Delta$. Also note that the EDM machine is a finite-memory approximation of the Cumulative Moving Average predictor given in Equation (2), where $\frac{1}{t+1}$ is replaced by the constant value $k^{-2/3}$ (which was shown to be optimal).

We now present asymptotic bounds on the regret achieved by the EDM machine when used to predict individual **continuous** sequences.

Theorem 2: The maximal regret of the k-states EDM machine (denoted U_{EDM_k}), attained by the worst continuous sequence, is asymptotically bounded by

$$\tfrac{1}{2}k^{-2/3} + O(k^{-1}) \leq \max_{x_1^n} R(U_{EDM_k}, x_1^n) \leq \tfrac{17}{4}k^{-2/3}$$

Proof: Consider L length sequence $\{x_t\}_{t=1}^{L}$ that endlessly rotates the machine in a minimal circle of L states $\{\hat{x}_t\}_{t=1}^{L}$. The input sample at each time t can be written as follows:

$$x_t = \hat{x}_t + (P_t\Delta + \delta_t)k^{2/3} \ , \tag{5}$$

where $P_t \in \mathbb{Z}$ denotes the number of states crossed by the machine at time t, δ_t is a quantization addition that satisfies $|\delta_t| < \frac{1}{2}\Delta$ and has no impact on the jump at time t, i.e. has no impact on the prediction at time $t+1$. Since we examine a minimal circle, the sum of states crossed on the way up is equal to the sum of states crossed on the way down, i.e $\sum_{t=1}^{L} P_t = 0$. By applying this and Jensen's inequality, the regret of the sequence can be upper bounded:

$$R(U_{EDM_k}, x_1^L) \leq \tfrac{1}{L}\sum_{t=1}^{L} \delta_t^2 k^{4/3} - \tfrac{1}{L}\sum_{t=1}^{L} 2P_t\Delta k^{2/3}\hat{x}_t \ . \tag{6}$$

The first term on the right hand side of Equation (6) depends only on the quantization of the input samples, δ_t, thus we term it *quantization loss*. The second term depends on the spacing gap between states, Δ, thus we term it *spacing loss*. Hence, the regret of the sequence is upper bounded by a loss incurred by the quantization of the input samples and a loss incurred by the quantization of the states' values, i.e. the prediction values. By applying $|\delta_t| < \frac{1}{2}\Delta$ we bound the *quantization loss*:

$$\textit{quantization loss} = \tfrac{1}{L}\sum_{t=1}^{L} \delta_t^2 k^{4/3} \leq \tfrac{1}{4}k^{-2/3} \ . \tag{7}$$

Now, let us upper bound the *spacing loss*. We define sub-step as a a single state step that is associated with a full step, e.g. a step of $P > 0$ states originated from state $\hat{x}$ consist P sub-steps, all associated with the origin state, $\hat{x}$. Since we examine minimal circle, it is possible to assign each down sub-step to an up-step that crosses the same state (an up-step of P_u states is assigned only with P_u down sub-steps). Noting that P_t is positive for up-steps and negative for down-steps, the following holds true:

$$\begin{aligned} -\tfrac{1}{L}\sum_{t=1}^{L} P_t\hat{x}_t &= -\tfrac{1}{L}\sum_{t\in\{\text{up steps}\}} P_t\hat{x}_t + \tfrac{1}{L}\sum_{t\in\{\text{down steps}\}} |P_t|\,\hat{x}_t \\ &= \tfrac{1}{L}\sum_{t\in\{\text{up steps}\}} \Big(- P_t\hat{x}_t + \sum_{j\in D(\hat{x}_t, P_t)} \hat{x}_j \Big) \ , \end{aligned} \tag{8}$$

where $D(\hat{x}_t, P_t)$ is the set of down sub-steps assigned to an up-step at time t. $\hat{x}_j$ is the origin state of sub-step j.

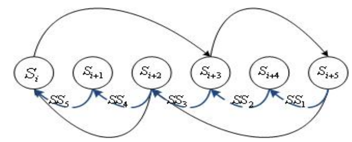

Fig. 2. Minimal circle of two up-steps and two down-steps (solid lines). SS_j are the down sub steps where $D(S_i, 3) = \{SS_3, SS_4, SS_5\}$, $D(S_{i+3}, 2) = \{SS_1, SS_2\}$. Note that sub steps SS_1, SS_2, SS_3 associated with origin state S_{i+5} while sub steps SS_4, SS_5 associated with origin state S_{i+2}.

Since the farthest up or down step in the EDM machine is $k^{-2/3}$, all down sub-steps in $D(\hat{x}_t, P_t)$ originated from a state that is not higher than $\hat{x}_t + P_t\Delta + k^{-2/3}$ and P_t can be bounded by $P_t \leq \frac{k^{-2/3}}{\Delta} \sim k^{1/3}$. Applying these and $\Delta \sim k^{-1}$ into Equation (8) results:

$$-\tfrac{1}{L}\sum_{t=1}^{L} P_t\hat{x}_t \leq \tfrac{1}{L}\sum_{t\in\{\text{up steps}\}} P_t(P_t\Delta + k^{-2/3}) \leq 2k^{-1/3} \ . \tag{9}$$

Thus, the *spacing loss* satisfies:

$$\textit{spacing loss} = 2\Delta k^{2/3}\big(-\tfrac{1}{L}\sum_{t=1}^{L} P_t\hat{x}_t\big) \leq 4k^{-2/3} \ . \tag{10}$$

By using Theorem 1, the upper bound is proven. The proof for the lower bound is given in Appendix I where we show that there is a sequence that endlessly rotates the k-states EDM machine in a minimal circle, incurring a regret of $\frac{1}{2}k^{-2/3} + O(k^{-1})$. ■

Note that Theorem 2 implies that the k-state EDM machine achieves regret smaller than $\frac{17}{4}k^{-2/3}$ for any individual continuous sequence. Moreover, the regret of the worst sequence is at least $\frac{1}{2}k^{-2/3} + O(k^{-1})$.

IV. Designing an optimal FSM with a small number of states

In the previous section we proposed the EDM machine as a universal predictor for continuous sequences and showed with an asymptotic analyze that it can achieve any vanishing regret with large enough number of states. However, what happens when only a small number of states are available? In this section we search for the best universal predictor with relatively small number of states. We start by presenting the optimal machines for a single, two and three states. The optimality is in sense of achieving the lowest maximal regret.

A. Single state universal predictor

The problem of finding the optimal single state machine has a trivial solution - from symmetry aspects, the optimal state is assigned with the value $\frac{1}{2}$ and the worst sequence, constant samples of 1 or 0, incurs a (maximal) regret of $R = \frac{1}{4}$.

B. Two states universal predictor

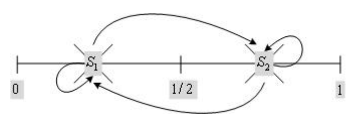

Fig. 3. Two states machine described geometrically over the $[0, 1]$ axis.

A two states machine has two possible minimal circles - zero-step circle (staying at the same state) and two steps circle (toggling between the two states). The lowest maximal regret is achieved when the regrets of both minimal circles are equalized. Thus, the lowest state is assigned with the value $S_1 = \sqrt{R}$ and a transition threshold $2\sqrt{R}$ and the second state with $S_2 = 1 - \sqrt{R}$ and a transition threshold $1 - 2\sqrt{R}$. Now, let us analyze the two steps minimal circle. Since the regret is convex with the input samples, there are only four possible sequences that can incur the highest regret:

$$x_1,\ x_2,\ x_1,\ x_2,\ \ldots$$

where $x_1 = 0$ or $x_1 = 1 - 2\sqrt{R}$ induces the down-step and $x_2 = 1$ or $x_2 = 2\sqrt{R}$ induces the up-step. By analyzing the regrets of these sequences one can note that the sequence $0, 1, 0, 1, \ldots$ incur the highest regret. Equalizing the regret incurred by this sequence to R results $R = (\frac{3}{8})^2$ and the optimal two states machine can be summarized:

- State values are:

$$S_1 = \frac{3}{8} \quad , \quad S_2 = \frac{5}{8}$$

- The states transition function satisfies:

$$\varphi(1, x) = \begin{cases} 1 & \text{if} \quad x < \frac{3}{4} \\ 2 & \text{otherwise} \end{cases}$$
$$\varphi(2, x) = \begin{cases} 1 & \text{if} \quad x < \frac{1}{4} \\ 2 & \text{otherwise} \end{cases}$$

  where $\varphi(i, x) = j$ is the transition function from state i to state j when the input sample is $x_t = x$.

The worst sequence that endlessly rotates the machine in one of the minimal circles incurs a (maximal) regret of $R = (\frac{3}{8})^2 = 0.14$.

Thus, if the desired regret is smaller than 0.14 we need to design a machine with more than two states.

C. Three states universal predictor

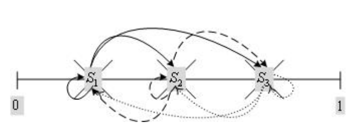

Fig. 4. Three states machine described geometrically over the $[0, 1]$ axis.

With the same considerations as for the two states machine, the lowest state is assigned with $S_1 = \sqrt{R}$ and the upper state with $S_3 = 1 - \sqrt{R}$. From symmetry aspects, the middle state is assigned with $S_2 = \frac{1}{2}$. We also note that if a two states jump is allowed from (to) the lower state to (from) the upper state, the sequence $0, 1, 0, 1, \ldots$ toggles the machine between these states, incurring a regret of $(\frac{3}{8})^2$. Hence, only a single state jump is allowed, otherwise the three states machine has no gain over the two states machine. Thus, in the same manner as for the two states machine, one can get that the optimal three states machine satisfies:

- State values are:

$$S_1 = 0.3285 \quad , \quad S_2 = 0.5000 \quad , \quad S_3 = 0.6715$$

- The states transition function satisfies:

$$\varphi(1, x) = \begin{cases} 1 & \text{if} \quad x < \ 0.6570 \\ 2 & \text{otherwise} \end{cases}$$
$$\varphi(2, x) = \begin{cases} 1 & \text{if} \quad x < \ 0.1715 \\ 2 & \text{if} \quad 0.1715 \leq \ x < \ 0.8285 \\ 3 & \text{otherwise} \end{cases}$$
$$\varphi(3, x) = \begin{cases} 2 & \text{if} \quad x < \ 0.3430 \\ 3 & \text{otherwise} \end{cases}$$

  where $\varphi(i, x) = j$ is the transition function from state i to state j when the input sample is $x_t = x$.

The worst sequence that endlessly rotates the machine in one of the minimal circles incurs a (maximal) regret of $R = 0.1079$.

Figure 5 depict the states and the transition thresholds over the $[0, 1]$ axis. One can notice the *hysteresis* characteristics of the machine, providing a "memory" or "inertia" to the finite-state predictor. An extreme input sample is needed for the machine to jump from the current state, that is, to change the prediction value.

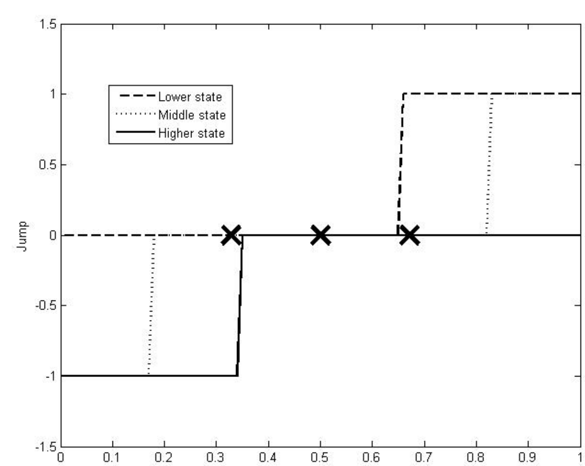

Fig. 5. Optimal three states machine described geometrically over the $[0, 1]$ axis along with the transition thresholds for each state. The X's represent the value assigned to each state.

D. The class of DTM machines

We now want to find a more general solution for the best universal predictor with a small number of states. We start by defining a new class of machines.

Definition 4: The class of all k-states ***Degenerated Tracking Memory*** (DTM) machines is of the form:

- An array of k states. We denote the states in the lower half $\{S_{k_l}, ..., S_1\}$ (in descending order where S_1 is the nearest state to $\frac{1}{2}$ and $S_i \leq \frac{1}{2}$ for all $1 \leq i \leq k_l$). We denote the states in the upper half $\{\bar{S}_1, ..., \bar{S}_{k_u}\}$ (in ascending order where $\bar{S}_1$ is the nearest state to $\frac{1}{2}$ and $\bar{S}_i > \frac{1}{2}$ for all $1 \leq i \leq k_u$), where $k_l + k_u = k$.
- The maximum down-step in the lower half, i.e. from states $\{S_{k_l}, ..., S_1\}$, is no more than a single state jump. The maximum up-step in the upper half, i.e. from states $\{\bar{S}_1, ..., \bar{S}_{k_u}\}$ is no more than a single state jump.
- A transition between the lower and upper halves is allowed only from and to the nearest states to $\frac{1}{2}$, S_1 and $\bar{S}_1$ (implying that the maximum up-jump (down-jump) from S_1 ($\bar{S}_1$) is a single state jump).

An example for a DTM machine is depict in Figure 6.

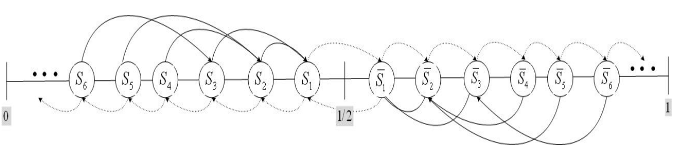

Fig. 6. An example of a DTM machine - note that a transition between the lower and upper halves is allowed only from (and to) S_1 and $\bar{S}_1$. Solid lines represent the maximum up or down jumps from each state.

In a DTM machine only a single state down-jump (up-jump) from all states in the lower (upper) half is allowed. In addition, the transition between the lower and upper halves is allowed only from and to the nearest states to $\frac{1}{2}$, S_1 and $\bar{S}_1$, implying that the maximum up-jump (down-jump) from $\{S_{k_l}, ..., S_2\}$ ($\{\bar{S}_2, ..., \bar{S}_{k_u}\}$) is up to S_1 ($\bar{S}_1$). These constraints facilitate the algorithm for constructing the optimal DTM machine.

E. Building the optimal DTM machine

We present here a schematic algorithm for constructing the optimal DTM machine. Given a desired regret, R_d, the task of finding the optimal DTM machine can be viewed as a covering problem, that is, assigning the smallest number of states in the interval $[0, 1]$, achieving a regret smaller than R_d for all sequences. We note that in an optimal k-state machine, the upper half of the states is the mirror image of the lower half. The symmetry property arises from the fact that any sequence $x_1, ..., x_n$ can be transformed into the symmetric sequence $1 - x_1, ..., 1 - x_n$. Both sequences achieve the same regret if full symmetry between the lower and upper halves is applied. Thus, assuming that the lower half is optimal in sense of achieving the desired regret with the smallest number of states, the upper half must be the reflection of the lower half to achieve optimality. Note that this property allows us to design the optimal DTM machine only for the lower half.

The algorithm we present here recursively finds the optimal states' allocation and their transition thresholds. Suppose states $\{S_{i-1}, ..., S_1\}$ in the lower half (in descending order where S_1 is the nearest state to $\frac{1}{2}$) and their transition thresholds set $\{\underline{T}_{\,i-1}, ..., \underline{T}_1\}$ are given and satisfying regret smaller than R_d for all minimal circles between them. Our algorithm generates the optimal S_i, i.e. the optimal allocation for state i, and a thresholds set, $\underline{T}_{\,i}$, satisfying regret smaller than R_d for all minimal circles starting at that state.

We start by finding S_1, the nearest state to $\frac{1}{2}$ in the lower half, in the optimal DTM machine.

Lemma 1: In the optimal k-states DTM machine for a given desired regret R_d, $S_1 = \frac{1}{2}$ if k is odd and

$$S_1 = \max\left\{1 - \sqrt{R_d + \frac{1}{4}}\ ,\ 2 + \sqrt{R_d} - 2\sqrt{R_d + \sqrt{R_d} + \frac{1}{2}}\right\}$$

if k is even.

Proof: From symmetry aspects $S_1 = \frac{1}{2}$ in the optimal DTM machine with odd number of states, otherwise there are more states in one of the halves and the symmetry property presented above does not hold. For even k, the nearest state to $\frac{1}{2}$ in the upper half, $\bar{S}_1$, is the mirror image of S_1, hence $\bar{S}_1 = 1 - S_1$. By definition, only a single state up-jump is allowed from S_1 and only a single state down-jump is allowed from $\bar{S}_1$. Thus, the machine can be rotated between these states, constructing a two steps minimal circle. Denote by x_1 and x_2 the samples that induce the up and down jumps, correspondingly. These samples must satisfy the transition thresholds, i.e.

$$\begin{aligned} S_1 + \sqrt{R_d} \leq\ & x_1\ \leq 1 \\ 0 \leq\ & x_2\ \leq \bar{S}_1 - \sqrt{R_d} = 1 - S_1 - \sqrt{R_d}\ . \end{aligned} \tag{11}$$

Since the regret is a convex function over the input samples, the regret of a minimal circle is brought to maximum by samples at the edges of the constraint regions. Thus, in a two steps minimal circle there are four combinations that may maximize the regret and need to be analyzed. Examining the regrets in all four cases, results that S_1 must satisfy two constraints $S_1 \geq 1 - \sqrt{R_d + \frac{1}{4}}$ and $S_1 \geq 2 + \sqrt{R_d} - 2\sqrt{R_d + \sqrt{R_d} + \frac{1}{2}}$. We choose the lowest S_1 that satisfy these constraints. ■

Note that S_1 must satisfy $S_1 \leq \frac{1}{2}$ which does not hold for low enough R_d, implying a lower bound on the achievable regret of the optimal DTM machine (see section IV-F).

Now, after presenting the starting state of the algorithm, we present the complete algorithm for constructing the optimal DTM machine:

1) *Set $i = 1$ and the corresponded starting state S_1 for odd or even number of states (see Lemma 1). Set the maximum up-step from the starting state $m_{u,1} = 1$.*
2) *Set the next state index $i = i + 1$.*
3) *For all $1 \leq m \leq i - 1$ (where m denotes the maximum up-step from state i) find the minimal $S_{i,m}$ with valid thresholds set $\underline{T}_{\,i,m}$ (in sequel we present the algorithm for finding the thresholds set).*
4) *Choose the minimal $S_{i,m}$ among all possible maximum up-steps, that is:*

$$m_{u,i} = \arg\min_{1 \leq m \leq i-1} S_{i,m}$$

$$S_i = S_{i,m_{u,i}}$$

$$\underline{T}_{\ i} = \underline{T}_{\ i,m_{u,i}}\ .$$

Thus we have set the parameters of state i: assigned value S_i, maximum up-jump of $m_{u,i}$ states and transition thresholds $\underline{T}_{\ i}$.

5) *If $S_i > \sqrt{R_d}$ go to step (2).*
6) *Set the upper half of the states to be the mirror image of the lower half.*

Explanations and Comments:

- For a given desired regret R_d, one should run the algorithm presented above twice - for odd and even number of states with the corresponded starting state, S_1. The optimal DTM machine is the one with the least states among the two (differ by a single state).
- Note that a transition thresholds for state 1 need to be given - a single state up-jump if the input sample satisfies $x \geq S_1 + \sqrt{R_d}$ and a single state down-jump if the input sample satisfies $x \leq S_1 - \sqrt{R_d}$. These are the optimal transition thresholds since as the interval for transition is wider the number of possible worst sequences in other minimal circles increases. Note that these transition thresholds achieve the maximal regret R_d for zero-step minimal circle (staying at S_1).
- A valid thresholds set for state i is a set of transition thresholds that satisfy regret smaller than R_d for all minimal circles starting at state i.

To complete the construction of the optimal DTM machine, we still need to present an algorithm for finding the optimal transition thresholds at each iteration (Step (3)). Consider states $\{S_{i-1}, ..., S_1\}$ in the lower half and their transition thresholds set $\{\underline{T}_{\ i-1}, ..., \underline{T}_1\}$ are given and satisfying regret smaller than R_d for all minimal circles between them. Suppose also S_i and m are given, where m denotes the maximum up-step from state i. Note that there are $m+1$ minimal circles starting at state i (depict in Figure 7):

- zero-step minimal circle (staying at state i).
- For any $2 \leq j \leq m+1$, a minimal circle of j steps - one up-step (of $j-1$ states), $j-1$ down-steps (of a single state).

Also note that these $m+1$ minimal circles are within the lower half, that is within the states $\{S_{i-1}, ..., S_1\}$ (since by definition a transition from the lower half to the upper half is allowed only from S_1).

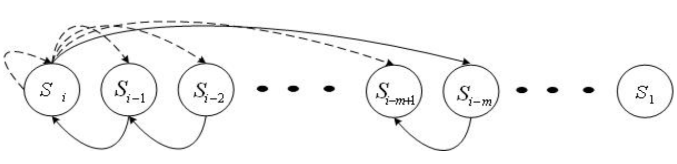

Fig. 7. $m+1$ possible minimal circles starting at S_i, where m is the maximum up-step from state i.

Let x_1^j be the samples that endlessly rotate the machine in a j steps minimal circle, where x_1 induces the up-step from state i and x_2^j induce the down-steps. Since the regret is convex in the input samples, the samples x_2^j that bring the regret to maximum are at the edges of the transition regions, that is, satisfying

$$x_t = \hat{x}_t - \sqrt{R_d} \quad \text{or} \quad x_t = 0 \quad \forall \quad 2 \leq t \leq \ j\ . \tag{12}$$

Suppose that for a given x_2^j, the regret of the sequence is smaller than R_d if x_1 satisfies

$$C_l(x_2^j) \leq x_1 \leq C_h(x_2^j)\ . \tag{13}$$

Thus, by Equation (12), x_1 must satisfy the constraint given in (13) for 2^{j-1} known combinations of x_2^j to satisfy regret smaller than R_d for any sequence that rotates the machine in this minimal circle. Hence,

$$\tilde{C}_l = \max_{x_2^j \in A_j} C_l(x_2^j) \leq x_1 \leq \min_{x_2^j \in A_j} C_h(x_2^j) = \tilde{C}_h \tag{14}$$

satisfies all the constrains, where A_j is the set of 2^{j-1} combinations of x_2^j according to Equation (12). Since x_1 must also satisfy the transition thresholds of state i, i.e.

$$T_{i,j-2} \leq x_1 \leq T_{i,j-1}\ , \tag{15}$$

we can conclude that the transition thresholds must satisfy

$$\begin{aligned} \tilde{C}_l &\leq T_{i,j-2}\ , \\ T_{i,j-1} &\leq \tilde{C}_h\ . \end{aligned} \tag{16}$$

Going over all minimal circles, $2 \leq j \leq m+1$, results bounds on all transition thresholds (upper and lower bound on each threshold). Thus, if a thresholds set can be found to satisfy all bounds and to cover the interval $[S_i + \sqrt{R_d}\ ,\ 1]$, we finished. Otherwise, no valid thresholds can be found for the given S_i and m.

Lemma 2: Consider a sequence x_1^j that rotates a DTM machine in a j steps minimal circle starting at state i. Given states $S_i, ..., S_{i-j+1}$, the regret is smaller than R_d if x_1 satisfies:

$$a(x_2^j) - b(x_2^j) \leq x_1 \leq a(x_2^j) + b(x_2^j)\ ,$$

where:

$$a(x_2^j) = S_i + \sum_{t=2}^{j} (S_i - x_t)\ ,$$

$$b(x_2^j) = j\sqrt{R_d - \frac{1}{j}\sum_{t=2}^{j} (S_{i-j+t-1} - S_i)(S_{i-j+t-1} + S_i - 2x_t)}\ . \tag{17}$$

Proof: Analyzing the regret of the sequence and claiming for regret smaller than R_d results the constrain on x_1:

$$\frac{1}{j}\sum_{t=1}^{j} [(x_t - \hat{x}_t)^2 - (x_t - \bar{x})^2] \leq R_d\ , \tag{18}$$

where $\hat{x}_1 = S_i$ and $\hat{x}_t = S_{i-j+t-1}$ for $2 \leq t \leq j$. ■

We can now present the algorithm for finding a thresholds set for state i given S_i and m, the maximum up-step:

1) *Find $C_{j,l}$ and $C_{j,h}$ for all $2 \leq j \leq m+1$ as follows:*

$$C_{j,l} = \max_{x_2^j \in A_j} \left\{ a(x_2^j) - b(x_2^j) \right\} ,$$
$$C_{j,h} = \min_{x_2^j \in A_j} \left\{ a(x_2^j) + b(x_2^j) \right\} , \quad (19)$$

where $a(x_2^j)$ and $b(x_2^j)$ are given in (17) *and A_j is the set of 2^{j-1} combinations of x_2^j:*

$$x_t = S_{i-j+t-1} - \sqrt{R_d} \quad or \quad x_t = 0 \quad \forall \quad 2 \leq t \leq \ j \ . \quad (20)$$

2) *If one of the following does not hold return and declare that there are no valid thresholds:*

$$\begin{aligned} &C_{j,l} < C_{j,h} \qquad \forall\ 2 \leq j \leq m \ , \\ &C_{j+1,l} \leq C_{j,h} \qquad \forall\ 2 \leq j \leq m \ , \\ &C_{2,l} \leq S_i + \sqrt{R_d} \ , \\ &1 < C_{m+1,h} \ . \end{aligned} \quad (21)$$

3) *Find a valid monotone increasing transition thresholds $\{T_{i,0}, \ldots, T_{i,m}\}$ that satisfy:*

$$\begin{aligned} &C_{j+1,l} \leq T_{i,j-1} \leq C_{j,h} \qquad \forall\ 2 \leq j \leq m \ , \\ &C_{2,l} \leq T_{i,0} \leq S_i + \sqrt{R_d} \ , \\ &1 < T_{i,m} \leq C_{m+1,h} \ . \end{aligned} \quad (22)$$

4) *Set the transition thresholds for the down-step* $\{0, S_i - \sqrt{R_d}\}$.

Explanations and Comments:

- $C_{j,l} < C_{j,h}$ must be satisfied otherwise there is no x_1 that satisfies regret smaller than R_d for all 2^{j-1} combinations of x_2^j.
- $C_{j+1,l} \leq C_{j,h}$ must be satisfied otherwise there is no $T_{i,j-1}$ satisfying both $T_{i,j-1} \leq C_{j,h}$ and $C_{j+1,l} \leq T_{i,j-1}$.
- $T_{i,0} \leq x_1 < T_{i,1}$ induces a single state up-jump, hence, $T_{i,0}$ must satisfy $C_{2,l} \leq T_{i,0}$. Also $T_{i,0}$ must satisfy $T_{i,0} \leq S_i + \sqrt{R_d}$ to ensure regret smaller than R_d for zero-step minimal circle (staying at state i).
- $T_{i,m-1} \leq x_1 < T_{i,m}$ induces m states up-jump, hence, $T_{i,m}$ must satisfy $T_{i,m} \leq C_{m+1,h}$. The transition thresholds must cover the interval $[S_i + \sqrt{R_d}, 1]$, therefore $T_{i,m}$ must also satisfy $1 < T_{i,m}$.
- This algorithm provides thresholds set given the states $\{S_{i-1}, ..., S_1\}$ and m, the maximum up-step from state i. It also requires S_i. Recalling the algorithm for finding S_i - we search for the minimal $S_{i,m}$ with a valid thresholds set for a given m. Thus, one can provide high $S_{i,m}$ and reduce it until no valid thresholds set can be found.

Theorem 3: The algorithm given in this section constructs the optimal DTM machine for a given desired regret, R_d, i.e. has the lowest number of states among all DTM machines that achieve regret R_d.

Proof: In each iteration the algorithm finds the minimal S_i with a valid thresholds set. Note that in DTM machines the transition thresholds for up-steps, $\{T_{i,0}, ..., T_{i,m_{u,i}}\}$, do not have an impact on regrets of minimal circles other than those starting at state i. Thus, given S_i, the optimality of these thresholds is only in sense of satisfying regret smaller than R_d for these minimal circles. As for the down thresholds - an input sample x induces a down-step from state s if satisfies $0 \leq x < T_{s,-1}$. As $T_{s,-1}$ is smaller for all states $s = i-1, ..., 1$ the achievable S_i with a valid thresholds set is smaller (the constrains are more relaxed). We choose the smallest $T_{s,-1}$ for all states, i.e. $S_s - \sqrt{R_d}$. Furthermore, each S_s is chosen to be minimal. We further show that optimality is achieved when assigning the minimal value for all states. Consider $\{S_{\lceil \frac{k}{2} \rceil}, ..., S_1\}$ in the lower half are the outputs of the algorithm for a given desired regret R_d. Let us examine the case where the assigned value for state $i-1$ is $\tilde{S}_{i-1}$ satisfying $\tilde{S}_{i-1} > S_{i-1}$. We note that the value assigned to state $i-1$ has no impact on the optimality of states $i-2, ..., 1$. Furthermore, the constrains on the up thresholds of state i depend only on $S_s - S_i$ or $S_s^2 - S_i^2$, where $s = i-1, ..., 1$ (applying $x_t = 0$ or $x_t = S_{i-j+t-1} - \sqrt{R_d}$ in Equation (17)). Since S_i is the minimal value with valid thresholds for $\{S_{i-1}, ..., S_1\}$, the minimal value with valid thresholds for $\{\tilde{S}_{i-1}, S_{i-2}, ..., S_1\}$ is not smaller than S_i. This holds for all states $\lceil \frac{k}{2} \rceil, ..., i$ and therefore, choosing $\tilde{S}_{i-1}$ does not reduce the number of states.

Thus, in all aspects optimality is achieved at each iteration in the algorithm by assigning state i with the minimal value S_i, down thresholds $\{0, S_i - \sqrt{R_d}\}$ and valid up thresholds. ■

F. Lower Bound - The Limitation of the DTM Machines

Theorem 4: The achievable regret of any DTM machine is lower bounded by

$$R = (\tfrac{1}{6})^2 = 0.0278 \ .$$

Proof: In an optimal k-states DTM machine, where k is even, the starting state S_1, must satisfies

$$S_1 = \max\{1 - \sqrt{R_d + \tfrac{1}{4}}, 2 + \sqrt{R_d} - 2\sqrt{R_d + \sqrt{R_d} + \tfrac{1}{2}}\} \leq \tfrac{1}{2}, \quad (23)$$

implying that if the desired regret satisfies $\sqrt{R_d} < \frac{1}{6}$, then $S_1 > \frac{1}{2}$ and no DTM machine with even number of states can be formed. We then conclude that also a DTM machine with odd number of states can not be formed otherwise a sub-optimal DTM machine with even number of states could have been formed by adding another state. ■

G. Numerical results

Figure 8 shows numerical results (number of states vs. regret) of the optimal DTM machine and the asymptotic EDM machine (regret of $\frac{1}{2}k^{-2/3}$). While the EDM machine can achieve any vanishing regret with large enough number of states, the lower bound for the DTM machine is depicted - as the number of states grows, the achievable regret goes to 0.0278.

We further note that the optimal DTM machine with a single, two and three states is identical to the optimal solution presented above for these machines, concluding that up to a certain number of states, our algorithm generates the optimal solution in sense of achieving the lowest maximal regret. Yet, it is unresolved up to which number of states.

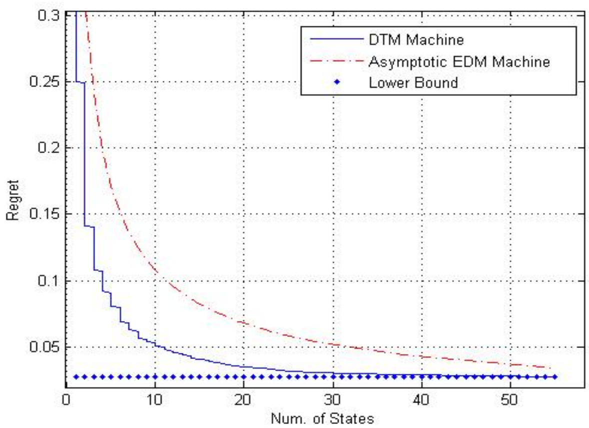

Fig. 8. The performance of the optimal DTM machine and the EDM machine.

V. Lower bound on the achievable regret of any k-states machine

In the previous section we have analyzed machines with a relatively small number of states. We shall now study the case of k-states machines, where k is relatively large. In section III we have proposed the EDM machine and showed that asymptotically, using enough states, it can achieve any vanishing regret. However, is it the optimal solution? (i.e. does it achieve a desired regret R with the lowest number of states?). In this section we present a lower bound on the number of states used by *any* machine that achieves a maximal regret R.

Definition 5: Given a starting state S_i, a ***Threshold Sequence x***, denoted $TS(x)$, is constructed for any x in the following manner - if the current state is smaller than x, next sample in the sequence is 1 (inducing an up-step), if not, next sample is 0 (inducing a down-step).

For any starting state and any x, the constructed $TS(x)$ induces a monotone jumps to the vicinity of x and than rotates the machine in a minimal circle. If the starting state is below x, the $TS(x)$ induces monotone up-steps until the machine crosses x (or monotone down-steps if the starting state is above x). In the vicinity of x the $TS(x)$ rotates the machine only in a bounded number of states - the lowest possible state is bounded from below by the maximum down-jump from the nearest state to x and the highest possible state is upper bounded by the maximum up-jump from the nearest state to x. Therefore, the $TS(x)$ endlessly rotates the machine in a finite number of states, thus inducing a minimal circle. We can conclude that for any x there is a starting state that the corresponded $TS(x)$ induces a minimal circle (removing the monotone jumps). As from now, we assume that any $TS(x)$ endlessly rotates the machine in a minimal circle without the monotone part.

Lemma 3: Consider a FSM that achieves a maximal regret R. A $TS(x)$ induces a minimal circle where at least half of its states are within $\frac{R}{x}$ from x for any $x \leq \frac{1}{2}$ and $\frac{R}{1-x}$ for any $x > \frac{1}{2}$.

Proof: Let us examine the regret of a $TS(x)$, where $x \leq \frac{1}{2}$, that rotates a FSM, denoted U, in a minimal circle of length L. Since the empirical mean of the sequence, $\bar{x}$, achieves the minimal square error, the regret satisfies:

$$\begin{aligned} R(U, x_1^L) &\geq \tfrac{1}{L} \sum_{t=1}^{L} (x_t - \hat{x}_t)^2 - (x_t - x)^2 \\ &\geq \tfrac{1}{L} \sum_{t=1}^{L} 2(x - \hat{x}_t)(x_t - x) \ . \end{aligned} \tag{24}$$

We note that by construction $(x - \hat{x}_t)(x_t - x)$ is positive for all t. Moreover, since $x \leq \frac{1}{2}$ and $x_t = 1$ for up-steps and $x_t = 0$ for down-steps, it follows that:

$$R(U, x_1^L) \geq \tfrac{1}{L} \sum_{t=1}^{L} 2\,|x - \hat{x}_t|\, x \ . \tag{25}$$

Hence half of the states have to be within $\frac{R}{x}$ from x, otherwise we get a regret higher than R. In the same manner it can be shown that for $x > \frac{1}{2}$ half of the states have to be within $\frac{R}{1-x}$ from x.

■

Lemma 4: Consider a FSM that achieves a maximal regret R. The maximum number of states crossed in an up-step and in a down-step from state S_i, for any i, must satisfy

$$m_{u,i} \geq \tfrac{1-(S_i+\sqrt{R})}{2\sqrt{R}}, \tag{26}$$

$$m_{d,i} \geq \tfrac{S_i-\sqrt{R}}{2\sqrt{R}} \ . \tag{27}$$

Proof: See Appendix II.

■

Note that Lemma 4 implies the same lower bound on the achievable regret of any DTM machine, $R \geq (\frac{1}{6})^2$ (as presented in section IV). Any DTM machine allows only a single state down-jump from all states below $\frac{1}{2}$. Thus, a DTM machine can achieve regret R if all states below $\frac{1}{2}$ satisfy Equation (27) with $m_{d,i} = 1$, hence:

$$\tfrac{\frac{1}{2}-\sqrt{R}}{2\sqrt{R}} \leq 1 \ . \tag{28}$$

Furthermore, Lemma 4 provides a lower bound on the maximal regret of any machine that allocates a state S_i with maximum up and down jumps of $m_{u,i}$ and $m_{d,i}$ states.

Theorem 5: The number of states in any deterministic FSM that achieves a regret smaller than R for any continuous sequence, is lower bounded by

$$\tfrac{1}{24} R^{-3/2} + O(R^{-1}) \ .$$

Proof: Consider a k-states machine that achieves a regret smaller than R for any sequence. Lemma 3 implies that for any $x \leq \frac{1}{2}$ there is a $TS(x)$ that forms a minimal circle in

the vicinity of x where at least half of the states are within $\frac{R}{x}$ from x. Since the samples of the $TS(x)$ are either 0 or 1, the constructed minimal circle is of at least $m_{u,i}$ states, where $m_{u,i}$ is the maximum up-jump from the nearest state to x, denoted state i. Thus, there are at least $\frac{1}{2}m_{u,i}$ states within $\frac{R}{x}$ from x. Lemma 4 implies that the maximum up-step from state i is at least $m_{u,i} = \lceil \frac{1-S_i-\sqrt{R}}{2\sqrt{R}} \rceil$ states, where S_i is the assigned value to state i.
We define the interval $B(m_u)$ as all x's satisfying:

$$m_u = \lceil \tfrac{1-x-\sqrt{R}}{2\sqrt{R}} \rceil \ . \tag{29}$$

Let us take:

$$\bigcup_{m_u \in \mathbb{N}} B(m_u) = [0, \tfrac{1}{2}] \ . \tag{30}$$

Using the fact that the minimal number of states in the lower half is equal to the minimal number of states in the upper half we can conclude that k, the number of states, satisfies:

$$\begin{aligned} k &\geq 2 \sum_{m_u \in \mathbb{N}} \min_{x \in B(m_u)} \tfrac{|B(m_u)|}{R/x} \tfrac{1}{2} m_u \\ &= \sum_{m_u \in \mathbb{N}} \min_{x \in B(m_u)} \tfrac{|B(m_u)|}{R/x} \lceil \tfrac{1-x-\sqrt{R}}{2\sqrt{R}} \rceil \\ &\geq \tfrac{1}{2} R^{-3/2} \sum_{m_u \in \mathbb{N}} \min_{x \in B(m_u)} |B(m_u)| \, x(1-x-\sqrt{R}) \ . \end{aligned} \tag{31}$$

Since $|B(m_u)| = 2\sqrt{R}$ for almost all m_u ($|B(m_u)| \leq 2\sqrt{R}$ at the edges of the interval $[0, \frac{1}{2}]$) and $x(1-x)$ is a concave function with a singular maximum point at $\frac{1}{2}$ we can conclude that choosing $x = \min\{x \in B(m_u)\}$ brings the most right hand side of Equation (31) to minimum, thus:

$$\begin{aligned} k &\geq \tfrac{1}{2} R^{-3/2} \sum_{i=1}^{\lfloor 1/(4\sqrt{R}) \rfloor} 2\sqrt{R}(\tfrac{1}{2} - 2\sqrt{R}i)(\tfrac{1}{2} + 2\sqrt{R}i - \sqrt{R}) \\ &\geq \tfrac{1}{24}(R^{-3/2} - 9R^{-1} - 4R^{-1/2}) \ . \end{aligned} \tag{32}$$

■

Note that Theorem 5 implies that a k-states FSM can not achieve a regret smaller than

$$(24k)^{-2/3} + O(k^{-1}) \tag{33}$$

for all sequences, i.e. the maximal regret is lower bounded by (33).

VI. Enhanced Exponential Decaying Memory machine

In this section we present a new FSM named the ***Enhanced Exponential Decaying Memory*** (E-EDM) machine, targeting to achieve any vanishing desired regret. We show that the E-EDM machine outperforms the performance of the EDM machine and approaches the lower bound presented in the previous section.

A. Designing the E-EDM machine

The algorithm for constructing the E-EDM machine for a desired regret, denoted R_d, is as follows:

- Set $R = \frac{R_d}{2}$.
- Divide the $[0,1]$ axis into segments, where a segment $A(m_u, m_d)$ is defined as the set of all x's satisfying:

$$\begin{aligned} m_u &= \lceil \tfrac{1-x-\sqrt{R}}{2\sqrt{R}} \rceil \quad \forall \quad x \in A(m_u, m_d) \ , \\ m_d &= \lceil \tfrac{x-\sqrt{R}}{2\sqrt{R}} \rceil \quad \forall \quad x \in A(m_u, m_d) \ . \end{aligned} \tag{34}$$

- According to Lemma 4, assign all states in segment $A(m_u, m_d)$ with maximum up and down jumps of m_u, m_d states, correspondingly.
- Linearly spread states in each segment $A(m_u, m_d)$ with a $\Delta(m_u, m_d)$ spacing gap between them where

$$\Delta(m_u, m_d) = \tfrac{\sqrt{R}}{2m_u \cdot m_d} \ . \tag{35}$$

- We further need to guarantee the desired regret when the machine traverses between segments. Consider two adjacent segments $A(m_{u,1}, m_{d,1})$ and $A(m_{u,2}, m_{d,2})$ and suppose the spacing gap in the second segment is smaller. Add states to the first segment such that the closest $(m_{u,1}+m_{d,1})$ states to the second segment have a spacing gap of $\Delta(m_{u,2}, m_{d,2})$. It can be shown that at most two states need to be added to each segment. Figure 9 depict the spacing gap in two adjacent segments.
- Assign transition thresholds for each state i as follows:

$$T_{i,j} = S_i + (2j+1)\sqrt{R} \quad \forall \quad -m_{d,i} \leq j \leq m_{u,i} \ , \tag{36}$$

that is, if the machine at time t is at state i, it jumps j states if the current outcome, x_t, satisfies:

$$S_i + (2j-1)\sqrt{R} \leq x_t < S_i + (2j+1)\sqrt{R} \ . \tag{37}$$

Note that as required, the transition thresholds cover the $[0,1]$ axis (arises from the chosen maximum up and down jumps).

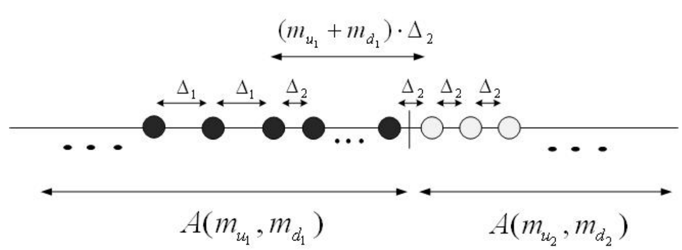

Fig. 9. Spacing gap of the E-EDM machine. Adjacent segments $A(m_{u,1}, m_{d,1})$ and $A(m_{u,2}, m_{d,2})$ with spacing gap $\Delta_s = \frac{\sqrt{R}}{2m_{u,s}m_{d,s}}$ where $s = 1, 2$ and $\Delta_2 < \Delta_1$. Note that the spacing gap between the highest $m_{u,1} + m_{d,1}$ states in segment $A(m_{u,1}, m_{d,1})$ is Δ_2 while the maximum up and down jumps from these states are $m_{u,1}$ and $m_{d,1}$ states.

Theorem 2 implies that the maximal regret of the k-states EDM machine is at least $\frac{1}{2}k^{-2/3}$. Note that if equality holds, the definitions of the EDM machine, excluding the part of allocating states, are identical to the definitions of the E-EDM machine. Thus, the new machine presented here can be regarded as an improvement of the EDM machine by better allocating the states - the states of the EDM are uniformly distributed over the interval $[0,1]$ while in the E-EDM machine the interval $[0,1]$ is divided into segments and states are uniformly distributed with a different spacing in each segment.

Theorem 6: The regret of the E-EDM machine is smaller than R_d for any input sequence.

Proof: Consider a sequence x_1^L that endlessly rotates the E-EDM machine (denoted U_{E-EDM}) in a minimal circle of L states $\hat{x}_1^L$. Each input sample x_t can be written as follows:

$$x_t = \hat{x}_t + 2\sqrt{R} \cdot P_t + \delta_t \ , \tag{38}$$

where P_t is the number of states the machine crosses at time t ($-m_d \leq P_t \leq m_u$) and δ_t satisfies $\delta_t \leq \sqrt{R}$ and can be regarded as a quantization addition that has no impact on the jump at time t, i.e. has no impact on the next prediction. Since we examine a minimal circle, the sum of states crossed on the way up is equal to the sum of states crossed on the way down, i.e $\sum_{t=1}^{L} P_t = 0$. By applying this and Jensen's inequality, the regret of the sequence satisfies:

$$R(U_{E-EDM}, x_1^L) \leq \frac{1}{L}\sum_{t=1}^{L}\delta_t^2 - 4\sqrt{R}\frac{1}{L}\sum_{t=1}^{L}P_t(\hat{x}_t - \hat{x}_1) \ . \tag{39}$$

We term the first loss in the right hand side of Equation (39) *quantization loss* (since it depends only on δ_t, the quantization of the input sample, x_t). By applying $\delta_t \leq \sqrt{R}$ we get:

$$\textit{quantization loss} = \frac{1}{L}\sum_{t=1}^{L}\delta_t^2 \leq R \ . \tag{40}$$

We term the second loss in the right hand side of Equation (39) *spacing loss* (since $\hat{x}_t - \hat{x}_1$ depends only on the spacing gap between states). Thus, as we sowed for the EDM machine, the regret of the sequence is upper bounded by a loss incurred by the quantization of the input samples and a loss incurred by the quantization of the states' values, i.e. the prediction values.

Lemma 5: For any sequence x_1^L that endlessly rotates the E-EDM machine in a minimal circle of states $\hat{x}_1^L$, where the spacing gap between all states is identical, the spacing loss is smaller than R satisfying:

$$\textit{spacing loss} = -4\sqrt{R}\frac{1}{L}\sum_{t=1}^{L}P_t(\hat{x}_t - \hat{x}_1) \leq R \ . \tag{41}$$

Proof: See Appendix III. ■

Lemma 6: For any sequence x_1^L that rotates the E-EDM machine in a minimal circle of states $\hat{x}_1^L$, where the spacing gap is not equal between all states, the spacing loss is smaller than R satisfying:

$$\textit{spacing loss} = -4\sqrt{R}\frac{1}{L}\sum_{t=1}^{L}P_t(\hat{x}_t - \hat{x}_1) \leq R \ .$$

Proof: See Appendix IV. ■

Since $R = \frac{R_d}{2}$ and by applying Theorem 1 we conclude that the E-EDM machine achieves a regret smaller than R_d for any sequence.

■

Theorem 7: The number of states in an E-EDM machine designed to achieve a regret smaller than R_d for all sequences is

$$\frac{1}{12}(\frac{R_d}{2})^{-3/2} + O(R_d^{-1}) \ .$$

Proof: See Appendix V. ■

B. Numerical results

Theorem 2 implies that the asymptotic worst regret of the k-states EDM machine is at least $\frac{1}{2}k^{-2/3}$. Thus, the number of states in an EDM machine that achieves a regret R_d, is at least $(2R_d)^{-3/2}$ states. Theorem 5 implies that the asymptotic number of states of any deterministic FSM that achieves a maximal regret R_d is at least $\frac{1}{24}R_d^{-3/2}$. Theorem 7 implies that the asymptotic number of states in an E-EDM machine that achieves a regret R_d is $\frac{1}{12}(\frac{R_d}{2})^{-3/2}$. Thus we can conclude that:

1) For a given desired regret, the E-EDM machine outperforms the EDM machine in number of states by a factor of:
$$\frac{\frac{2^{3/2}}{12}R_d^{-3/2}}{(2R_d)^{-3/2}} = \frac{2}{3} \ ,$$
i.e. uses only $\frac{2}{3}$ of the states needed for the EDM machine to achieve the same maximal regret.
2) For a given desired regret, the E-EDM machine approaches the lower bound with a factor of about:
$$\frac{\frac{2^{3/2}}{12}R_d^{-3/2}}{\frac{1}{24}R_d^{-3/2}} = 2^{5/2} = 5.6 \ .$$

Simulation results are presented in Figure 10. Note that the theoretical results match the numerical results and show that for a large number of states the E-EDM machine outperforms the EDM machine by a factor of $\sim \frac{2}{3}$ and approaches the lower bound with a factor of ~ 6.

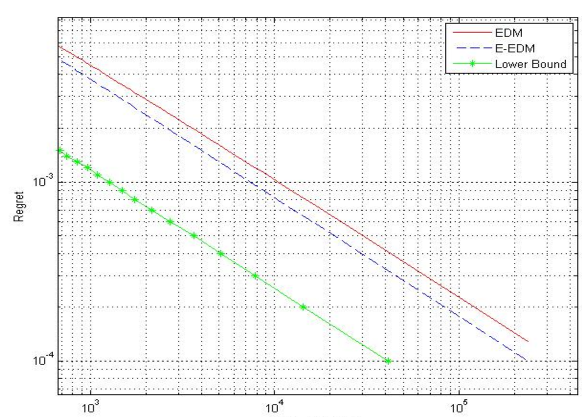

Fig. 10. Comparing the performance of the E-EDM machine, the EDM machine and the lower bound.

VII. SUMMARY AND CONCLUSIONS

In this paper we studied the problem of universal least squares prediction of individual continuous-alphabet sequences when limited resources are available.

For universal predictors with a small number of states, or equivalently for high allowable regret, we presented the optimal Degenerated Tracking Memory (DTM) machine, which performs well with a small number of states yet its achievable regret is lower bounded by $R = 0.0278$. Numerical results showed that the optimal DTM machine indeed outperforms any other machine for a small enough number of states. However, it is still unknown up to which number of states it is the best universal predictor. For number of states larger than that, one can try to attain better performance by easing the constrains of the DTM machines and allowing more than a single state down-jump (up-jump) from all states in the lower (upper) half. However, the construction of the optimal machine in this case is much more complex.

For universal predictors with a large number of states, or equivalently for any vanishing desired regret, we proved a lower bound of $O(k^{-2/3})$ on the achievable regret of any k-state machine. We proposed the Exponential Decaying Memory (EDM) machine and showed that the worst sequence incurs a bounded regret of $O(k^{-2/3})$, where k is the number of states. We further presented the Enhanced Exponential Decaying Memory (E-EDM) machine which outperforms the EDM machine. The E-EDM machine can be regarded as an improvement of the EDM machine by better allocating the states over the interval $[0, 1]$. Recalling that the EDM machine is a finite-memory approximation of the Cumulative Moving Average predictor which is the best unlimited resources universal predictor (w.r.t the non-universal empirical mean predictor), we can understand why both the EDM and the E-EDM machines approach optimal performance.

Analyzing the performance of the EDM and the E-EDM machines showed that the regret of any sequence can be upper bounded by the sum of two losses - *quantization loss*, the loss incurred by the quantization of the input samples, and *spacing loss*, the loss incurred by the quantization of the prediction values. It is worth mentioning that the worst regret of the optimal DTM machine can also be upper bounded by the sum of these losses. As the number of states in the optimal DTM machine increases, the *quantization loss* goes to the lower bound, $R = 0.0278$, and the *spacing loss* goes to zero. Thus, understanding the optimal allocation between these two losses may lead to the answer of up to which number of states the optimal DTM machine is the best universal predictor. It is also worth mentioning that the E-EDM machine is constructed with allocating half of the desired regret to the *quantization loss* and the other half to the *spacing loss*. A further optimization may be obtained by a different allocation.

Throughout this paper we assumed that the sequence's outcomes are bounded. Note that this constraint is mandatory. Since the performance is analyzed by the worst sequence, there is no universal predictor which attains a finite regret for unbounded sequences (a sequence which incur an infinite regret can always be found). However, an optional further study is to expand the results we achieved to a more relaxed case, e.g. sequences where the difference between consecutive outcomes is bounded.

In this study we essentially examined finite-memory universal predictors trying to attain the performance of the (non-universal) "zero-order", constant predictor, i.e. trying to attain the empirical variance of any individual continuous sequence. We believe that our work is the first step in the search for the best finite-memory universal predictor trying to attain the performance of the best (non-universal) L-order predictor, for any L.

APPENDIX I
PROOF OF THE LOWER BOUND GIVEN IN THEOREM 2

Proof: Here we show that there is a continuous-valued sequence which rotates the EDM machine (denoted U_{EDM}) in a minimal circle incurring a regret of $\frac{1}{2}k^{-2/3} + O(k^{-1})$.

Consider the following minimal circle - m states up-step, $m-1$ states down-step, m states up-step, $m-1$ states down-step and so on $m-1$ times. The last step is a down-step of $m-1$ states that close the circle and return the machine to the initial state. Denoting the states' gap by Δ, the described sequence can be written as follows:

$$
\begin{aligned}
x_1 &= \hat{x}_1 + (m + \tfrac{1}{2} - \xi)\Delta k^{2/3} \\
x_2 &= \hat{x}_1 + m\Delta - (m - 1 - \tfrac{1}{2} + \xi)\Delta k^{2/3} \\
x_3 &= \hat{x}_1 + \Delta + (m + \tfrac{1}{2} - \xi)\Delta k^{2/3} \\
&\vdots \\
x_{2m-3} &= \hat{x}_1 + (m-2)\Delta + (m + \tfrac{1}{2} - \xi)\Delta k^{2/3} \\
x_{2m-2} &= \hat{x}_1 + (2m-2)\Delta - (m - 1 - \tfrac{1}{2} + \xi)\Delta k^{2/3} \\
x_{2m-1} &= \hat{x}_1 + (m-1)\Delta - (m - 1 - \tfrac{1}{2} + \xi)\Delta k^{2/3}
\end{aligned}
$$

where $\xi \to 0$.

Analyzing the regret of the described sequence results:

$$
\begin{aligned}
R(U_{EDM}, x_1^{2m-1}) &= \tfrac{1}{4}\Delta^2 k^{4/3} + m(m-1)\Delta^2 k^{2/3} - \tfrac{m(m-1)}{3}\Delta^2 \\
&= \tfrac{1}{4}k^{-2/3} + m(m-1)k^{-4/3} - \tfrac{m(m-1)}{3}k^{-2}\ .
\end{aligned} \tag{42}
$$

Note that in the vicinity of $\frac{1}{2}$ an input sample 1 or 0 induces an up or down step (accordingly) of $\frac{1}{2}k^{-2/3}$. Therefore we can choose:

$$
m = \frac{\frac{1}{2}k^{-2/3}}{\Delta} \sim \frac{1}{2}k^{1/3}\ . \tag{43}
$$

We further note that there is $\hat{x}_1$ for which all samples are valid, meaning all samples satisfy $0 \leq x_t \leq 1$. For example: $\hat{x}_1 = \frac{1}{2} - \frac{1}{2}k^{-1/3} - \frac{1}{2}k^{-2/3}$.

Now, applying Equation (43) into Equation (42) results:

$$
\begin{aligned}
regret &= \tfrac{1}{2}k^{-2/3} - \tfrac{1}{2}k^{-1} - \tfrac{1}{12}k^{-4/3} + \tfrac{1}{6}k^{-5/3} \\
&= \tfrac{1}{2}k^{-2/3} + O(k^{-1})\ .
\end{aligned} \tag{44}
$$

■

APPENDIX II
PROOF OF LEMMA 4

Proof: Consider a sequence $x_1, ..., x_{L+1}$ that rotates a FSM, denoted U, in a minimal circle, where x_1 induces a single up-jump of L states and x_2^{L+1} induce down-jumps of a single state. Since the regret of any zero-step minimal circle is smaller than R, an input sample that satisfies $x = \hat{x}_t - \sqrt{R} - \varepsilon$, where $\varepsilon \to 0^+$, must induce a down-jump of at least one state. Thus, we can always choose the input samples x_2^{L+1} to satisfies $x_t \geq \hat{x}_t - \sqrt{R}$. We shall also assume that x_1 satisfies:

$$x_1 > \hat{x}_1 + (1+2L)\sqrt{R} \ , \tag{45}$$

where $\hat{x}_1 = S_i$. ***We show that this assumption can not hold true***.

By denoting $\lambda_t = \hat{x}_t - \hat{x}_1$ we note that the empirical mean of the sequence satisfies:

$$\bar{x} \geq \hat{x}_1 + \sqrt{R} + \tfrac{1}{L+1} \sum_{t=1}^{L+1} \lambda_t \ . \tag{46}$$

Now, let us examine the regret incurred by the described sequence:

$$\begin{aligned} R(U, x_1^L) &= \tfrac{1}{L+1} \sum_{t=1}^{L+1} (x_t - \hat{x}_t)^2 - (x_t - \bar{x})^2 \\ &= (\bar{x} - \hat{x}_1)^2 + \tfrac{1}{L+1} \sum_{t=1}^{L+1} \lambda_t^2 - 2\lambda_t (x_t - \hat{x}_1) \\ &\geq (\bar{x} - \hat{x}_1)^2 - \tfrac{1}{L+1} \sum_{t=1}^{L+1} \lambda_t^2 && (47) \\ &\geq (\sqrt{R} + \tfrac{1}{L+1} \sum_{t=1}^{L+1} \lambda_t)^2 - \tfrac{1}{L+1} \sum_{t=1}^{L+1} \lambda_t^2 && (48) \\ &> R + \tfrac{1}{L+1} \sum_{t=1}^{L+1} (2\sqrt{R} - \lambda_t)\lambda_t \ , && (49) \end{aligned}$$

where (47) follows $\lambda_t \geq 0$ and $x_t \leq \hat{x}_t$ for all the down samples x_2^{L+1}, (48) follows (46). In [13] it is shown that in a FSM that achieves a maximal regret R w.r.t binary sequences, the maximal up-jump is no more than $2\sqrt{R}$. Therefore, this must hold also for continuous-valued sequences. Hence, in the discussed minimal circle all states are within $2\sqrt{R}$ from the initial state, that is $2\sqrt{R} \geq \lambda_t$ for all t and we get $R(U, x_1^L) > R$.

We can now conclude that to achieve a regret smaller than R, any input sample x that induces an L states up-jump from state i, must satisfy:

$$x \leq S_i + (1+2L)\sqrt{R} \ . \tag{50}$$

Since an input sample 1 induces an $m_{u,i}$ states jump from state i we conclude that the following must be satisfied:

$$1 \leq S_i + (1+2m_{u,i})\sqrt{R} \ . \tag{51}$$

In the same manner it can be shown that $0 \geq S_i - (1 + 2m_{d,i})\sqrt{R}$.

■

APPENDIX III
PROOF OF LEMMA 5

Proof: First we note that:

$$-\tfrac{1}{L} \sum_{t=1}^{L} P_t (\hat{x}_t - \hat{x}_1) = -\tfrac{1}{L} \sum_{t=1}^{L} P_t \hat{x}_t \ , \tag{52}$$

where we used $\sum_{t=1}^{L} P_t = 0$. Note that $P_t \hat{x}_t$ is positive for up-steps and negative for down-steps. We consider a minimal circle within a segment $A(m_u, m_d)$ that crosses states with the same spacing gap, denoted $\Delta = \Delta(m_u, m_d)$. It follows that:

$$-\tfrac{1}{L} \sum_{t=1}^{L} P_t (\hat{x}_t - \hat{x}_1) = -\tfrac{1}{L} \sum_{t=1}^{L} P_t \sum_{j=1}^{t-1} P_j \Delta \ .$$

Define ***mixed*** sequences as sequences where the up and down steps are interlaced. Define ***straight*** sequences as sequences where all the up-steps are first, followed by all the down-steps (consecutive in time). We show that any ***mixed*** sequence with $\{P_t\}_{t=1}^L$ jumps that rotates the machine in a minimal circle with the same spacing gap for all states can be transformed into a ***straight*** sequence with the same jumps only in a different order (up-jumps are first) without changing the *spacing loss* of the sequence. First we note that for any three interlaced jumps

up jump $\to$ down jump $\to$ up jump,

that cross

$$P_{u,1} \ \to \ P_d \ \to \ P_{u,2}$$

states (accordingly), the following holds true:

$$\begin{aligned} &P_{u,1}\hat{x}_{u,1} + P_d(\hat{x}_{u,1} + P_{u,1}\Delta) + \\ &\qquad + P_{u,2}(\hat{x}_{u,1} + (P_{u,1} + P_d)\Delta) \\ &\quad = P_{u,1}\hat{x}_{u,1} + P_{u,2}(\hat{x}_{u,1} + \\ &\qquad + P_{u,1}\Delta) + P_d(\hat{x}_{u,1} + (P_{u,1} + P_{u,2})\Delta) \ . \end{aligned} \tag{53}$$

Thus, Equation (53) implies that the *spacing loss* of these three jump does not change when the order of the jumps is:

up jump $\to$ up jump $\to$ down jump.

This can be shown also for a sequence with more than one consecutive down-jumps between two up-steps:

up jump $\to$ down jump $\to$ ... $\to$ down jump $\to$ up jump .

Hence, in a recursive way any ***mixed*** sequence can be transformed into a ***straight*** sequence without changing the *spacing loss* by moving all the down-jumps to the end of the sequence. In the rest of the proof we shall assume ***straight*** sequences. Note that this transformation changes the states of the minimal circle, but since we transform the sequence only for an easier analyze, we can assume that all states still have the same spacing gap. Figure 11 gives an example.

We continue by proving that applying maximum up and down steps maximize the *spacing loss*. Consider two consecutive down-steps of P_{d_1}, P_{d_2} states staring at state $\hat{x}$, with a total of C states, i.e $|P_{d_1}| + |P_{d_2}| = C$. Note that we examine

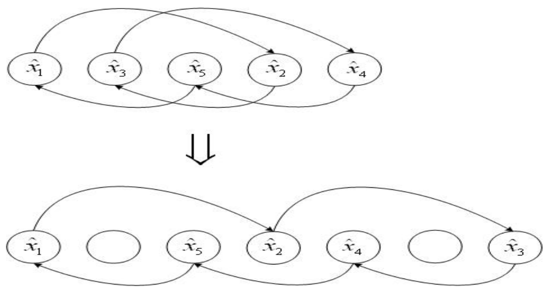

Fig. 11. An example for a ***mixed*** sequence transformed into a ***straight*** sequence.

two down-steps, thus $C \le 2m_d$. The *spacing loss* of these two down-steps is:

$$\hat{x}\cdot|P_{d,1}|+(\hat{x}-|P_{d,1}|\,\Delta)\cdot|P_{d,2}| = \hat{x}\cdot C-|P_{d,1}|\,(C-|P_{d,1}|)\Delta \ . \tag{54}$$

If $C \le m_d$ the *spacing loss* is maximized for $|P_{d,1}| = C$ and $|P_{d,2}| = 0$. If $m_d \le C \le 2m_d$ then the *spacing loss* is maximized for $|P_{d,1}| = m_d$. We got that we can maximize the *spacing loss* by taking a couple of down-steps and unite them into a single down-step (if together they cross no more than m_d states), or to apply maximum down-step, m_d, to the first and $C-m_d$ to the second (if together they cross more than m_d states). Thus, assuming ***straight*** sequences, we can start with the first couple of down-steps, maximize the *spacing loss* by applying maximum down-step, then take the third down-step and apply maximum down-step with the new down-steps that were created. In a recursive way we can maximize the *spacing loss* by applying maximum down-steps (note that the number of down-steps reduces which also maximize the *spacing loss*). In the same manner it can be shown that applying maximum up-steps maximize the *spacing loss*.

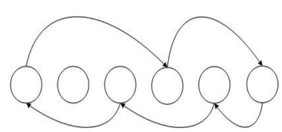

Fig. 12. An example for the worst case *spacing loss* of a minimal circle that crosses 5 states in the segment $A(3,2)$.

Consider a minimal circle of C states crossed on the way up and down, all in the segment $A(m_u, m_d)$. The worst case scenario for the *spacing loss* is composed of N_u up-steps each of m_u states jump (maximum up-jump), a single up-step of c_u states, where $c_u = mod(C, m_u)$, N_d down-steps each of m_d states jump (maximum down-jump), and a single down-step of c_d states, where $c_d = mod(C, m_d)$. N_d and N_u satisfy $C = N_u m_u + c_u$ and $C = N_d m_d + c_d$. It can be shown that the position in the sequence of the single up-step (of c_u states) and the single down-step (of c_d states) has no impact on the *spacing loss*. Let us analyze the *spacing loss* of the *straight* sequence. *First, all up-steps satisfy:*

$$\begin{aligned}
-\tfrac{1}{L} \sum_{t\in\{\textit{up steps}\}} & P_t(\hat{x}_t-\hat{x}_1) = \\
&= -\tfrac{1}{L}\Delta(\sum_{i=0}^{N_u-1} m_u(i\cdot m_u) + N_u m_u c_u) \\
&= -\tfrac{1}{L}\Delta(m_u^2\tfrac{N_u(N_u-1)}{2} + N_u m_u c_u) \\
&= -\tfrac{1}{L}\tfrac{\Delta}{2}(C^2 - m_u C + c_u(m_u-c_u)) \ .
\end{aligned} \tag{55}$$

In the same manner, all down-steps satisfy:

$$\begin{aligned}
-\tfrac{1}{L} \sum_{t\in\{\textit{down steps}\}} & P_t(\hat{x}_t-\hat{x}_1) = \\
&= \tfrac{1}{L}\Delta(\sum_{i=1}^{N_d} m_d(i\cdot m_d) + c_d C) \\
&= \tfrac{1}{L}\tfrac{\Delta}{2}(C^2 + m_d C - c_d(m_d-c_d)) \ .
\end{aligned} \tag{56}$$

Thus, the worst case scenario of the spacing loss *satisfies:*

$$-\tfrac{1}{L}\sum_{t=1}^{L} P_t(\hat{x}_t-\hat{x}_1) =$$

$$= \tfrac{1}{L}\tfrac{\Delta}{2}(C(m_u+m_d) - c_u(m_u-c_u) - c_d(m_d-c_d)) \tag{57}$$

$$\le \tfrac{1}{L}\tfrac{\Delta}{2}C(m_u+m_d) \ , \tag{58}$$

where the length of the circle satisfies:

$$L = \lceil \tfrac{C}{m_u} \rceil + \lceil \tfrac{C}{m_d} \rceil \ge \tfrac{C}{m_u} + \tfrac{C}{m_d} \ . \tag{59}$$

Therefore, the worst case scenario satisfies:

$$-\tfrac{1}{L}\sum_{t=1}^{L} P_t(\hat{x}_t-\hat{x}_1) \le \tfrac{m_u m_d}{2}\Delta \ . \tag{60}$$

Since $\Delta = \Delta(m_u, m_d) = \frac{\sqrt{R}}{2m_u m_d}$ *we get that the* spacing loss *for any minimal circle within a segment (and with identical spacing gap between all states) satisfies:*

$$\text{spacing loss} \le 4\sqrt{R}\tfrac{m_u m_d}{2}\Delta(m_u, m_d) = R \ . \tag{61}$$

■

Appendix IV
Proof of Lemma 6

Proof: We denote two adjacent segments by $A(m_{u,1}, m_{d,1})$ and $A(m_{u,2}, m_{d,2})$. Assume $A(m_{u,1}, m_{d,1})$ is the lower segment and the minimal circle starts at the lowest state. Denote the spacing gap of each segment by $\Delta_1 = \Delta(m_{u,1}, m_{d,1})$ and $\Delta_2 = \Delta(m_{u,2}, m_{d,2})$. Note that if $\Delta_1 < \Delta_2$ then $m_{u,2} = m_{u,1} - 1$, $m_{d,2} = m_{d,1}$ and if $\Delta_1 > \Delta_2$ then $m_{u,2} = m_{u,1}$, $m_{d,2} - 1 = m_{d,1}$.

First we assume that the minimal circle traverse between the segments only once (that is, once on the way up and once on the way down). We also assume that $\Delta_1 < \Delta_2$. We can now divide the minimal circle into two virtual minimal circles - take the up-step that traverse the machine to the higher segment and denote the destination state of this jump by $\hat{x}_c$. Take a down-step that crosses state $\hat{x}_c$ and split it into two steps - assuming the down-step crosses P_d states, c_d states jump to

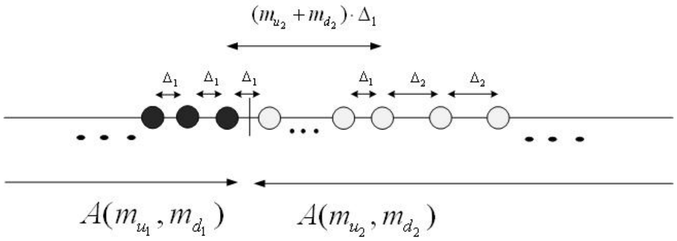

Fig. 13. Spacing gap between states in the connection between the segments $A(m_{u,1},m_{d,1})$ and $A(m_{u,2},m_{d,2})$. See the E-EDM machine definitions in section VI.

state $\hat{x}_c$ and (P_d-c_d) states jump from state $\hat{x}_c$. Note that two minimal circles were constructed - left minimal circle that traverse C_1 states and right minimal circle that traverse C_2 states. This is depict in Figure 14. The *spacing loss* of the down-step satisfies:

$$P_d(\hat{x}_c+c_d\Delta_1)=c_d(\hat{x}_c+c_d\Delta_1)+(P_d-c_d)\hat{x}_c+(P_d-c_d)c_d\Delta_1\ . \tag{62}$$

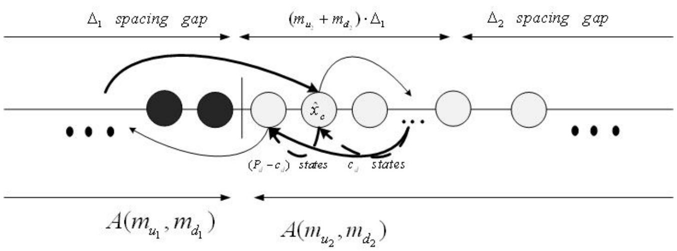

Fig. 14. Minimal circle that traverse once between segments. Splitting the marked down-step that crosses state $\hat{x}_c$ into two down-steps, creating two virtual minimal circles to the right and left. Note that since the first $m_{u,2}+m_{d,2}$ states at the second segment are with spacing gap Δ_1, the marked down-step must only cross states with spacing gap Δ_1.

Note that $\hat{x}_c$ is in the upper segment but we used Δ_1 since the first $m_{u,2}+m_{d,2}$ states in the upper segment have spacing gap of Δ_1 (see the construction of the E-EDM machine in section VI-A). Also note that the first term in the right hand side of Equation (62) belongs to the *spacing loss* of the right minimal circle and the middle term belongs to the *spacing loss* of the left minimal circle. Note that the *spacing loss* of the minimal circle is compose of the *spacing loss* of the left and right minimal circles and the last term in Equation (62). The left minimal circle traverse C_1 states, all with spacing gap Δ_1. The right minimal circle traverse C_2 states, some with spacing gap Δ_1 and some with Δ_2. We can now conclude that the *spacing loss* satisfies:

$$\begin{aligned}\textit{spacing loss} \le 4\sqrt{R}\tfrac{1}{L}\Big(\ &[C_1(m_{u,1}+m_{d,1})\\ &-(P_d-c_d)(m_{d,1}-(P_d-c_d))]\tfrac{\Delta_1}{2}\\ &+[C_2(m_{u,2}+m_{d,2})-c_d(m_{d,2}-c_d)]\tfrac{\Delta_2}{2}\\ &+c_d(P_d-c_d)\Delta_1\ \Big)\ ,\end{aligned} \tag{63}$$

where we applied Lemma 5 (Equation (57)) to bound the *spacing loss* of the left and right minimal circles. Note that Lemma 5 is true for the right minimal circle since all states have a spacing gap that is no more than Δ_2. Now, since $m_{d,1}=m_{d,2}$ and $\Delta_1<\Delta_2$ we get:

$$\begin{aligned}\textit{spacing loss} &\le 4\sqrt{R}\tfrac{1}{L}(C_1(m_{u,1}+m_{d,1})\tfrac{\Delta_1}{2}+\\ &\qquad +C_2(m_{u,2}+m_{d,2})\tfrac{\Delta_2}{2})\\ &=R\tfrac{1}{L}(\tfrac{C_1}{m_{d,1}}+\tfrac{C_1}{m_{u,1}}+\tfrac{C_2}{m_{d,2}}+\tfrac{C_2}{m_{u,2}})\ .\end{aligned} \tag{64}$$

Let us bound the length of the minimal circle:

$$\begin{aligned}L &\ge \lceil\tfrac{C_1}{m_{u,1}}\rceil+\lceil\tfrac{C_2}{m_{u,2}}\rceil+\lceil\tfrac{C_1+C_2}{m_{d,1}}\rceil\\ &\ge \tfrac{C_1}{m_{u,1}}+\tfrac{C_2}{m_{u,2}}+\tfrac{C_1+C_2}{m_{d,1}}\ .\end{aligned} \tag{65}$$

Applying this into Equation (64) results:

$$\textit{spacing loss} \le R\ . \tag{66}$$

Assume again that the minimal circle traverse between the segments only once but now assume $\Delta_1>\Delta_2$. Divide the minimal circle into two virtual minimal circles in the same manner as above but now take the down-step that traverse the machine to the lower segment and split an up-step. In the same manner we can show that the *spacing loss* is not more than R.

If assuming that the minimal circle traverse between segments m times, in the same manner as above we can divide the circle into m left minimal circles and m right minimal circles and bound the *spacing loss*.

■

Appendix V
Proof of Theorem 7

Proof: Consider an E-EDM machine that was designed to achieve regret R_d. By denoting $R=\frac{R_d}{2}$, the number of states satisfies:

$$k \le \sum_{m_u,m_d\in\mathbb{N}}\big(\tfrac{|A(m_u,m_d)|}{\Delta(m_u,m_d)}+2\big)\ , \tag{67}$$

where all states in the segment $A(m_u,m_d)$ have a maximum up and down step of $m_u,\ m_d$ states and $\Delta(m_u,m_d)$ spacing gap. As shown in the definitions of the E-EDM machine in section VI, we add to each segment at most two states to ensure regret smaller than R_d for sequences that rotate the E-EDM machine in a minimal circle that traverse between segments. Note that there are at most $\lceil\frac{1}{2\sqrt{R}}\rceil$ segments.
Let us examine Equation (67):

$$\begin{aligned}k &\le R^{-1/2}+2+\sum_{m_u,m_d\in\mathbb{N}}\tfrac{|A(m_u,m_d)|}{\Delta(m_u,m_d)}\\ &=R^{-1/2}+2+\sum_{m_u,m_d\in\mathbb{N}}\tfrac{|A(m_u,m_d)|}{\sqrt{R}}2m_um_d\\ &=R^{-1/2}+2+2R^{-1/2}\sum_{m_u,m_d\in\mathbb{N}}|A(m_u,m_d)|\cdot\\ &\qquad\cdot\lceil\tfrac{1-x-\sqrt{R}}{2\sqrt{R}}\rceil\cdot\lceil\tfrac{x-\sqrt{R}}{2\sqrt{R}}\rceil\Big|_{x\in A(m_u,m_d)}\\ &\le R^{-1/2}+2+\tfrac{1}{2}R^{-3/2}\sum_{m_u,m_d\in\mathbb{N}}|A(m_u,m_d)|\cdot\\ &\qquad\cdot\big(x(1-x)+\sqrt{R}+R\big)\Big|_{x\in A(m_u,m_d)}\ .\end{aligned} \tag{68}$$

By denoting the segments with the same maximum up-step as $B(m_u)$, we can further bound the number of states:

$$k \leq \tfrac{1}{2}(R^{-1}+3R^{-1/2})+2+\tfrac{1}{2}R^{-3/2}\sum_{m_u\in\mathbb{N}}|B(m_u)|\cdot \cdot \max_{x\in B(m_u)} x(1-x) \ . \quad (69)$$

Since $|B(m_u)| = 2\sqrt{R}$ for almost all m_u ($|B(m_u)| \leq 2\sqrt{R}$ at the edges of the interval $[0,\frac{1}{2}]$), $x(1-x)$ is a concave function with a singular maximum point at $\frac{1}{2}$ and the number of states in the lower and upper halves is equal, we get:

$$\begin{aligned} k &\leq \tfrac{1}{2}\big(R^{-1}+3R^{-1/2}\big)+2+ \\ &\quad + R^{-3/2}\sum_{i=1}^{\lceil\frac{1}{4\sqrt{R}}\rceil} 2\sqrt{R}(\sqrt{R}+i2\sqrt{R})(1-(\sqrt{R}+i2\sqrt{R})) \\ &\leq \tfrac{1}{12}R^{-3/2}-\tfrac{5}{12}R^{-1}-12R^{-1/2}-32 \\ &= \tfrac{2^{3/2}}{12}R_d^{-3/2}+O(R_d^{-1}) \ , \end{aligned} \quad (70)$$

where we applied $R=\frac{R_d}{2}$.

We can also bound the number of states from below by:

$$\begin{aligned} k &\geq \sum_{m_u,m_d\in\mathbb{N}} \tfrac{|A(m_u,m_d)|}{\Delta(m_u,m_d)} \\ &\geq \tfrac{1}{2}R^{-3/2}\sum_{m_u,m_d\in\mathbb{N}}|A(m_u,m_d)|\cdot\big(x(1-x)- \\ &\qquad -\sqrt{R}+R\big)\Big|_{x\in A(m_u,m_d)} \ . \end{aligned} \quad (71)$$

By denoting the segments with the same maximum up-step as $B(m_u)$, we can bound the number of states from below:

$$\begin{aligned} k \geq \tfrac{1}{2}\big(&-R^{-1}+R^{-1/2}+ \\ &+R^{-3/2}\sum_{m_u\in\mathbb{N}}|B(m_u)|\cdot\min_{x\in B(m_u)}x(1-x)\big) \ . \end{aligned} \quad (72)$$

Using the approximation we made to calculate the lower bound we get:

$$\begin{aligned} k &\geq \tfrac{1}{12}(R^{-3/2}-15R^{-1}+2R^{-1/2}) \\ &= \tfrac{1}{12}(\tfrac{R_d}{2})^{-3/2}+O(R_d^{-1}) \ . \end{aligned} \quad (73)$$

Thus, we upper and lower bounded the number of states in the E-EDM machine by $\frac{1}{12}(\frac{R_d}{2})^{-3/2}+O(R_d^{-1})$. ■

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